

Q IN OTHER SOLAR SYSTEMS

ARISTOTLE SOCRATES*, BOAZ KATZ*[†] & SUBO DONG^{††}
Institute for Advanced Study, Princeton, NJ 08540, USA
Draft version September 26, 2012

ABSTRACT

A significant fraction of the hot Jupiters with final circularized orbital periods of $\lesssim 5$ days are thought to form through the channel of high-eccentricity migration. Tidal dissipation at successive periastron passages removes orbital energy of the planet, which has the potential for changes in semi-major axis of a factor of ten to a thousand. In the equilibrium tide approximation we show that, in order for high-eccentricity migration to take place, the relative level of tidal dissipation in Jupiter analogues must be at least 10 times higher than the upper-limit attributed to the Jupiter-Io interaction. While this is not a severe problem for high- e migration, it contradicts the results of several previous calculations. We show that these calculations of high- e migration inadvertently over-estimated the strength of tidal dissipation by three to four orders of magnitude. These discrepancies were obscured by the use of various parameters, such as lag time τ , tidal quality factor Q and viscous time t_V . We provide the values of these parameters required for the Jupiter-Io interaction, tidal circularization and high- e migration. Implications for tidal theory as well as models of the inflated radii of hot Jupiters are discussed. Though the tidal Q is not, in general, well-defined, we derive a formula for it during high-eccentricity migration where Q is approximately constant throughout evolution.

Subject headings:

1. INTRODUCTION

The recent explosion in the discovery of extrasolar planets has re-invigorated interest in theory of tidal excitation and dissipation. The proximity of the hot Jupiters to the stellar host suggests that tidal forces shaped or are currently shaping the orbital and structural properties of these close-in planets.

High- e migration (e.g., Rasio & Ford 1996; Wu & Murray 2003; Fabrycky & Tremaine; see Socrates et al. (2011) for a summary) is a class of migration scenarios that depend upon the excitation of eccentricity, e , to high values where tidal dissipation is activated as small periastron distances. Successive periastron passages convert orbital energy into heat that is ultimately radiated away. The discovery that the orbits of a significant fraction of the hot Jupiters are mis-aligned with the spin of their stellar hosts lends strength to the claim that some high- e migration is at work (Winn et al. 2010).

For such mechanisms to work, tidal dissipation during periastron passage must be strong enough to dissipate the required amount of orbital energy within the age of the system, which for FGK stars on the main sequence is a significant fraction of the Hubble time. This requirement places a lower bound on the strength of dissipation, which can then be tested against theories of the tidal excitation and dissipation. By utilizing equilibrium tidal theory in the weak friction approximation, Goldreich & Soter (1966; hereafter GS) produced the first estimate of the level of dissipation in gas giants (Jupiter and Saturn), parameterized by a tidal quality factor Q .

There is a general understanding in the literature that high- e migration is a viable mechanism for forming hot Jupiters in the event that the relative level of tidal dissipation, within the context of equilibrium theory, is consistent with that inferred by GS's study of the Jupiter-Io interaction. Here we critically re-examine the viability

of high- e migration and tidal circularization for hot extrasolar gas giant planets. We do so within the context of the equilibrium tidal theory of Hut (1981), which is widely employed in the study of high- e migration (e.g., Eggleton et al. 1998; Wu & Murray 2003; Fabrycky & Tremaine 2007; Naoz et al. 2011).

In §2 we summarize results from equilibrium tide theory, in the limit of weak dissipation, that are relevant for this work. We revisit the concept of the tidal quality factor Q and examine its relation to the time lag τ of the equilibrium tidal response. The required τ for high- e migration is estimated in §3. There we show that the required strength of tidal dissipation for high- e migration is at least 10 times stronger than the inferred upper-limit for dissipation arising from the Jupiter-Io interaction. In §4 we briefly review some previous calculations of high- e extrasolar planet migration and comment on their prescriptions for tidal dissipation. We also discuss §5 some specific dynamical tide models in the literature and show that they cannot account for circularization of the hot Jupiters and thus are unlikely account for their presumed high- e migration. We summarize in §6.

2. THEORY OF EQUILIBRIUM TIDES

2.1. *Physical Basis for Equilibrium Tides*

Hut (1981) asserts that the most attractive feature of the constant lag time model for tidal dissipation is its simplicity. That is, it is straightforward to implement in analyses of secular orbital dynamics. However, Socrates & Katz (2012) show that a constant single time lag τ follows directly from the equations of motion, given the following basic assumptions

- the tidal forcing and dissipation mechanism are slow and non-resonant¹

¹ Here, 'non-resonant' refers to the fluid response of the forced body.

- the equilibrium structure of the forced body is spherically symmetric
- the dissipation is weak such that the decay time of the forced response is long in comparison to the characteristic forcing periods
- only the quadrupolar component of the tidal force is important
- non-linear effects are unimportant

Altogether, these conditions lead to a constant lag time τ of the forced response, which is an intrinsic property of the tidally forced object in question. Therefore, if Jupiter were to be placed in another system where the characteristic forcing frequency and amplitude are different than that felt by Jupiter from Io, the lag time τ nevertheless remains the same as long as the assumptions above are correct.

Socrates & Katz (2012) show that, given the assumptions listed above, the constant lag time prescription follows directly from the equations of motion for a spherically symmetric forced fluid body. In the Appendix we show that, within the equilibrium tide approximation, quantification of the level of tidal dissipation with a tidal quality factor Q is only possible under the assumption of a single forcing frequency and a single constant lag time.

2.2. relevant results of equilibrium tides

Under the simplifying assumption of spin-orbit alignment, the amount of energy dissipated per orbit is given by

$$\begin{aligned} \Delta E &= - \oint_{\text{orb}} dt dE/dt \\ &= E_F \left(\frac{P_F}{t_D} \right) \times [A - 2Bx + Cx^2] \end{aligned} \quad (1)$$

and the change in specific orbital angular momentum per orbit is

$$\begin{aligned} \Delta J &= \oint_{\text{orb}} dt dJ/dt \\ &= -J \left(\frac{P_F}{t_D} \right) \times [B - xC] \end{aligned} \quad (2)$$

where $E_F = \frac{GM(M + m_{\text{per}})/a_F}{\sqrt{G(M + m_{\text{per}})a_F}}$, $J = \frac{2\pi/\sqrt{G(M + m_{\text{per}})a_F^3}}{2\pi/n_F}$ and $a_F \equiv a(1 - e^2)$. Here $x = \Omega/n_F$ where Ω is the rotation rate and $A - C$ are functions of e and given by eqs. A33 - A35 of Hut (1981). In the event of pseudo synchronous rotation where $\Omega = \Omega_{\text{ps}} = n_F B/C$, orbital angular momentum J is conserved and a_F is a constant and equal to the final circularization radius. Here M is the mass of the forced body, m_{per} is the mass of the perturber and

$$t_D \equiv \frac{M a_F^8}{3 k_L \tau G m_{\text{per}}^2 R^5} \quad (3)$$

is a characteristic time scale for tidal dissipation, where τ is the lag time, k_L is the Love number and R is the radius of the forced body.

The evolutionary equations for the semi-major axis a and eccentricity e are obtained by dividing eqs. (1) and

(2) by the orbital periods and then using Kepler's laws to relate the orbital energy E and J to e and a . The evolutionary equations in three limiting cases of interest are provided below.

2.2.1. synchronization: $e = 0$, $\Omega \neq \Omega_{\text{ps}}$, $M \gg m_{\text{per}}$

Spin down of a major body by a minor body leads to expansion of the orbital separation between the two as a result of angular momentum conservation. The rate of this expansion is given by

$$\frac{\dot{a}}{a} = - \frac{2(n - \Omega)}{n t_D} \frac{M}{m_{\text{per}}} \quad (4)$$

where n is the mean motion. The expression above determines the rate of orbital expansion e.g., of the Moon from the Earth. The ultimate source of energy and angular momentum originate from the spin of the major body.

2.2.2. circularization: $e \ll 1$, $\Omega = \Omega_{\text{ps}}$, $M \ll m_{\text{per}}$

When $e \ll 1$, the pseudosynchronous spin rate $\Omega_{\text{ps}} \approx n$ and the rate of decay of orbital eccentricity follows

$$\frac{\dot{e}}{e} = - \frac{2}{t_D}. \quad (5)$$

When the orbital angular momentum is large compared to the spin angular momentum, the forced body attains a pseudo synchronous state before circularization can proceed. The source of energy for dissipation during migration is primarily orbital.

2.2.3. high- e migration: $e \rightarrow 1$, $\Omega = \Omega_{\text{ps}}$, $M \ll m_{\text{per}}$

As in the case of circularization the spin is pseudo synchronous during high- e migration and therefore, the orbital angular momentum J remains a constant during inward migration. Consequently, the shape of the orbit near periastron is approximately constant as long as $e \approx 1$ and therefore, the energy dissipated per orbit ΔE (e.g., Socrates et al. 2012) is approximately a constant as well.

In the limit that $e \rightarrow 1$ the migration rate is given by

$$\frac{\dot{a}}{a} = - \frac{2007}{64 t_D} \sqrt{\frac{a_F}{a}} \quad (6)$$

which is consistent with the expectation that $\Delta E \approx a$ constant per orbit.

2.3. $Q \propto \tau$

Since Goldreich & Soter's (1966) measurement of the tidal quality factor Q of Jupiter, Q has become a common parameterization for the level of tidal dissipation in gas giant planets and stars. In what follows, we discuss the relationship and applicability of Q and the lag time τ .

In physics, the quality factor Q is often utilized to parameterize the level of dissipation in a forced damped harmonic oscillator that obeys

$$\ddot{X} + \omega^2 X + \gamma \dot{X} = F(t) \quad (7)$$

with $F = \sqrt{\frac{1}{2}} F_0 e^{i\sigma t} + \text{c.c.}$ The fundamental frequency ω and damping rate γ are intrinsic properties of the oscillator. When the forcing is slow such that $\sigma \ll \omega$ and

if the dissipation is weak, then the leading order solution $X^{(0)}$ is given by

$$X^{(0)} = \frac{F(t)}{\omega^2} \quad (8)$$

with respect to the tidal problem, the expression above is analogous to the equilibrium tide. The term $\propto \ddot{X}$ is small in comparison to the $X^{(0)}$ and does not contribute to energy transfer. To next order in forcing frequency $\sigma = 2\pi/P$ the contribution to X is due dissipation and is given by

$$X^{(1)} = -\gamma \dot{X}^{(0)} = -\tau \dot{F} \quad (9)$$

which allows us to write

$$X = \frac{1}{\sqrt{2}} \frac{F_0}{\omega^2} e^{i\sigma t - i\delta} + \text{c.c.} \quad (10)$$

where the time lag is given by

$$\tau = \gamma/\omega^2 \quad (11)$$

and the phase lag $\delta = \sigma\tau$.

For a damped, driven harmonic oscillator Q is defined as

$$\frac{1}{Q} = \frac{-\oint_{\text{cyc}} dt dE/dt}{2\pi E_0} = \frac{-\Delta E_{\text{cyc}}}{2\pi E_0} \quad (12)$$

where ΔE_{cyc} is the energy dissipated per forcing cycle and is given by

$$-\Delta E_{\text{cyc}} = \oint_{\text{cyc}} dt \dot{X} F \simeq 2\pi \frac{|F_0|^2}{\omega^2} \delta \quad (13)$$

and the peak energy E_0 , is the energy stored in the interaction

$$E_0 = \langle X F \rangle \equiv \frac{1}{P} \oint_{\text{cyc}} dt X F = \frac{|F_0|^2}{\omega^2} \quad (14)$$

and therefore,

$$\frac{1}{Q} = -\frac{\oint_{\text{cyc}} dt \dot{X} F}{2\pi \langle X F \rangle} = \delta = \sigma\tau. \quad (15)$$

The oscillator Q is used as a proxy for the phase lag of the forced response. Its definition requires that the time-dependence of the forcing be described by a single Fourier harmonic. Given the forcing frequency σ , a constant lag time τ is also required to define Q .

In general, the forced damped harmonic oscillator is not driven by a single Fourier harmonic and the time-dependence can be completely arbitrary. The more general solution in the non-resonant weak friction limit may be written as

$$\begin{aligned} X(t) &= X^{(0)} + X^{(1)} = X^{(0)} - \tau \dot{X}^{(0)} \\ &\simeq X^{(0)}(t - \tau) \end{aligned} \quad (16)$$

in the limit that τ is small and thus the response $X(t)$ follows the forcing $F(t)$ by some time lag τ . Note that the lag time τ parameterizes the level of dissipation for a damped harmonic oscillator, for arbitrary forcing. The oscillator Q , which is equivalent to the phase lag δ , characterizes the level of dissipation for the specialized case of sinusoidal forcing. Unlike ω and τ , Q is not an intrinsic property of the oscillator as it depends on a particular

property of the external driving force i.e., the forcing frequency σ .

As previously mentioned, Socrates & Katz (2012) show that for the tidal problem, a constant lag time τ is an intrinsic property of the forced body, given the physical requirements listed in §2.1, assuming the that fluid response of the forced body obeys the following equation of motion

$$\ddot{\boldsymbol{\xi}} + \mathbf{C} \cdot \dot{\boldsymbol{\xi}} + \mathbf{D} \cdot \boldsymbol{\xi} = -\nabla U_T \quad (17)$$

where $\boldsymbol{\xi}$ is the Lagrangian displacement and U_T is the tidal potential. The operators \mathbf{C} and \mathbf{D} can be thought of as a generalized “spring” coefficient and damping rate, respectively.

The tidal potential can be decomposed into multipoles of spherical harmonic degree ℓ . The lag time τ_ℓ of the corresponding tidal multipole response $q_{\ell m}$ is given by

$$\tau_\ell = \frac{\int d^3x \rho \boldsymbol{\Xi}_{\ell m}^* \cdot \mathbf{C}^{-1} \cdot \mathbf{D} \cdot \mathbf{C}^{-1} \cdot \boldsymbol{\Xi}_{\ell m}}{\int d^3x \rho \boldsymbol{\Xi}_{\ell m}^* \cdot \mathbf{C}^{-1} \cdot \boldsymbol{\Xi}_{\ell m}} \quad (18)$$

where $\boldsymbol{\Xi}_{\ell m}(\mathbf{x}) = \nabla r^\ell Y_{\ell m}(\boldsymbol{\Omega})$ describes the spatial variation of the tidal field about the co-ordinates, $\mathbf{x} = (r, \boldsymbol{\Omega})$, of the forced body. Both \mathbf{C} and \mathbf{D} are time-independent differential operators that depend, like the fluid density ρ , on the internal structure of the forced body. It then follows that, as in the case of the non-resonant weakly damped forced harmonic oscillator, the lag time is an intrinsic property of a non-resonant weakly damped tidally forced body.

Socrates & Katz (2012) also show that there is a constant time lag τ_ℓ for each spherical harmonic degree ℓ of the forced response. By adopting the approximation that the perturber is distant such that only the $\ell = 2$ terms are kept, they are then able to reproduce the basic assumptions of Hut (1981).

Even in the event that the time dependence of the tidal forcing is described by a single Fourier harmonic, it is not clear why, and under what conditions, the level of dissipation of the tidal problem can be parameterized by a quality factor Q . As previously mentioned, the tidal potential is the sum of the tidal harmonics, each of which may be a sum of Fourier harmonics. Furthermore, and on a more basic level, the left hand side of eq. 17 is not, in general, equivalent to the left hand of eq. 7. In other words, the tidal problem is not equivalent, in general, to a forced damped harmonic oscillator.

Nevertheless, in the limiting cases where the constant time lag model is valid and when there is a single forcing frequency, σ , the tidal quality factor Q is given by the ratio of the energy dissipated during the tidal forcing cycle and the average tidal interaction energy of the oscillating components given by (see Appendix)

$$\frac{1}{Q} = -\frac{\sum_{\ell=2,m} \oint_{\text{cyc}} \dot{q}_{\ell m}^* \Psi_{\ell m}}{\sum_{\text{osc}, \ell=2,m} 2\pi \langle q_{\ell m}^* \Psi_{\ell m} \rangle} = \delta = \sigma\tau. \quad (19)$$

here $q_{\ell m}$ is the induced multipolar moment arising from the presence of the (normalized) tidal potential $\Psi_{\ell m}$ and the \sum_{osc} in denominator enforces the inclusion of only the oscillating components of $q_{\ell m}$ and $\Psi_{\ell m}$. Note the correspondence between the expression above and the expression for the oscillator Q given by eq. 15. In the Appendix we show that, under several limiting conditions, the tidal Q as defined above is equivalent to a single phase lag.

2.3.1. $e=0$; synchronization

Both the $m = \pm 2$ components of the tidal potential oscillate sinusoidally at the same frequency $\sigma = 2|\Omega - n|$. The $m = 0$ perturbation is time-independent in this orbital configuration and therefore, does not contribute to secular evolution. It follows that the tidal Q is well defined as discussed above. As we state in the Appendix, the peak energy E_0 is due to equal contributions of both components. Furthermore, since the lag time τ is independent of m , the forced response corresponding to each component of the tidal forcing possess the same phase lag (and angle) as well and therefore, the same Q .

By using ΔE from eq. 1 and E_0 from eq. A11 we may write the familiar relation

$$\frac{1}{Q_{\text{synch}}} = 2 |\Omega - n| \tau. \quad (20)$$

In the expression above we made use of the fact the orbital period and the period of the forcing cycle differ by a ratio of $2|\Omega - n|/n$ in order to relate ΔE_{cyc} to ΔE of eq. 1.

2.3.2. $e \ll 1$; circularization

Again, the tidal Q is well defined for this orbital configuration in part because the tidal forcing is described by a single forcing frequency $\sigma = n$. The forcing is composed of three distinct Fourier harmonics that originate from the $m = 0$ and $m = \pm 2$ components of the tidal potential. As we state in the Appendix, the peak energy E_0 is the sum of all three components, where the contributions from the $m = 0$ and $m = \pm 2$ components are not equal. Furthermore, since the lag time τ is independent of m , the forced response corresponding to each component of the tidal forcing possesses the same phase lag (and angle) as well and therefore, the same Q . Consequently, the level of dissipation can be parameterized by a single Q as well.

By using ΔE from eq. 1 in the limit that $\Omega = \Omega_{\text{ps}}$ (cf. eq. A 36 of Hut (1981)) and along with the expression for the peak energy E_0 from eq. A14 we may write

$$\frac{1}{Q_{\text{circ}}} = n \tau. \quad (21)$$

Here, ΔE of eq. 1 is equal to ΔE_{cyc} since the forcing frequency is equal to the mean motion n .

2.3.3. $e \approx 1$; high- e migration

In this limit, Q is not well defined i.e., there is neither a well-defined forcing cycle in order to compute an \oint_{cyc} -type integral nor a well-defined phase lag since the forcing is not sinusoidal. Nevertheless, it is possible to define a forcing cycle that is *approximately periodic* in the limit that $e \rightarrow 1$ since tidal potential and thus \dot{E} vanishes at apoastron. In order to define a Q for high- e migration a peak energy E_0 is needed, which we set to the peak interaction given by

$$E_0 = \frac{k_L G m R^5}{r_p^6} \quad (22)$$

where $r_p = a(1 - e)$ is the periastron distance. Now we may define a tidal Q during high- e migration

$$\frac{1}{Q_{\text{HEM}}} \equiv -\frac{\Delta E}{2\pi E_0} = \frac{3861}{4096} n_F \tau. \quad (23)$$

TABLE 1
INFERRED VALUES OF VARIOUS TIDAL DISSIPATION
PARAMETERS FOR JUPITER-LIKE PLANETS ^a

parameters	Jupiter-Io interaction	hot Jupiter circularization	high- e migration ^b
τ (s)	0.062	0.25	0.66
Q	5.9×10^4	2.8×10^5	1.1×10^5
t_D (Gyr)	9.5×10^5	20	7.5
T (hrs)	1.2×10^4	3.0×10^3	1.1×10^3
t_V (hrs)	1.3×10^5	3.3×10^4	1.2×10^4
$\sigma_V^c (10^{-59})$	6.3	25	67

^aWe take the Love number $k_L = 0.38$, $R_p = 6.99 \times 10^9$ cm, $M_p = 1.9 \times 10^{30}$ g, $m_{\text{Io}} = 8.93 \times 10^{25}$ g, the mass of the stellar companion = M_\odot and the final circularization distance $a_F = 0.06$ AU.

^bFor high- e migration the tidal Q is not well-defined and we use Q_{HEM} as defined by eq. 23.

^cThe parameter σ_V has dimensions of $(\text{g cm}^2 \text{s})^{-1}$.

Interestingly, Q_{HEM} as defined above remains approximately a constant during high- e migration. Take for example, a case where the initial value of eccentricity is $e = 0.999$ and tidal dissipation shrinks the orbit to a value of $e = 0.99$. Though the orbital period changes by a factor of ~ 30 , the value of Q_{HEM} changes by less than 1%. The shape of the orbit near periastron – where all of the energy transfer takes place – remains approximately constant during evolution (e.g., Socrates et al. 2012). It follows that $\Delta E_{\text{cyc}} = \Delta E$ per orbit remains approximately a constant as does the peak energy E_0 and therefore, so should Q_{HEM} .

2.4. other parameterizations

Within the literature, there are other parameterizations of the constant lag time for various theoretically-based motivations. Expressions relating them to the lag time τ are given below.

Hut (1981) defines a characteristic time scale for dissipation

$$T \equiv \frac{R^3}{G M \tau} \quad (24)$$

Eggleton et al. (1998; see also Fabrycky & Tremaine 2007) use the viscous time t_V

$$t_V = 3(1 + 1/k_L) T = 3(1 + 1/k_L) \frac{R^3}{G M \tau} \quad (25)$$

and the coefficient σ_V incorporated by Hansen (2010; 2012)

$$\sigma_V = \frac{2}{3} k_L \tau G R^{-5}. \quad (26)$$

3. APPLICATION TO GAS GIANT PLANETS

In what follows, we determine the values of τ , t_V , and σ_V for Jupiter, based on its interaction with Io. We then compare these values with the required values for circularization and high- e migration of Jupiter analogues ($M = M_J$, $R = R_J$ and same structure). In particular, we estimate the value of Q and t_D for these cases.

3.1. Jupiter-Io interaction

Jupiter's rotation is super-synchronous with respect to its orbit with Io ($\Omega_J > n_{\text{Io}}$). In order to obtain a timescale for the evolution of Io's orbit, use eq. 4 and integrate from $a = 0$ to $a = a_{\text{Io}}$ in order to write

$$\frac{a}{\dot{a}}|_{a=a_{\text{Io}}} = \frac{13}{2\Lambda} t_{\odot} \quad (27)$$

where $t_{\odot} = 4.5$ Gyr is the age of the Sun and

$$\Lambda = \frac{\sum_{i=\text{Io}}^{\text{Ganymede}} L_i}{L_{\text{Io}}} \approx 4.3 \quad (28)$$

where L_i is the orbital angular momentum if the i^{th} Galilean moon. In obtaining an upper limit for dissipation, the inclusion of the factor Λ conservatively assumes that the current resonant configuration of the inner three Galilean moons is preserved during outward migration. From the form of t_D given by eq. 3, we obtain a limit of the lag time $\tau_{\text{J-Io}}$ for the Jupiter-Io interaction

$$\tau_{\text{J-Io}} \leq 0.062 \text{ s} \quad (29)$$

where we have set $k_L = 0.38$. By using eq. 20 we may determine the lower-limit for $Q_{\text{J-Io}}$, the tidal Q for the Jupiter-Io interaction

$$Q_{\text{J-Io}} \geq 5.9 \times 10^4 \quad (30)$$

as found by other authors (Goldreich & Soter 1966; Yoder & Peale 1981; Leconte et al. 2010).

3.2. circularization of hot Jupiters

Nearly all hot Jupiters with orbital periods $P \leq 5$ days and semi-major axes $a \leq 0.06$ AU have circular orbits, while the values of e belong to a broad distribution for gas giants at larger separation. A common interpretation is that planets that are relatively close-in circularize quickly due to tidal dissipation's strong dependance on orbital separation. Consequently, it is possible to obtain a separate constraint on the dissipation parameters τ , Q , etc., by presuming that tidal dissipation is responsible for circularizing Jupiter-like planets with $P \leq 5$ days within a characteristic age $t_0 = 10$ Gyrs of these systems.

If we set $t_0^{-1} = \dot{e}/e$, given by the right hand side of eq. 5 then we can write an expression for t_D

$$t_D = 2 t_0. \quad (31)$$

With the help of eq. 3 and by setting $k_L = 0.38$ and $a_F = 0.06$ AU we arrive at a constraint for τ

$$\tau \geq 0.25 \text{ s} \quad (32)$$

for a Jupiter-analogue in orbit around a Sun-like star. In the above constraint, we chose $k_2 = 0.38$ and $a_F = 0.06$ AU. Given this value of τ , eq. 21 enables us to determine Q

$$Q_{\text{circ}} \leq 2.8 \times 10^5. \quad (33)$$

3.3. high- e migration of hot Jupiters

A leading theory for the formation of hot Jupiters is high- e migration (see Socrates et al. 2012 for a review). In particular Fabrycky & Tremaine (2007) predicted that, as a result of high- e migration, the orbital angular momentum of hot Jupiters should, in general, be mis-aligned with the spin axis of the stellar host,

which was later confirmed observationally by Winn et al. (2010).

The migration time t_m between an initial eccentricity e_i and some final value e_0 can be found by numerically integrating eq. 6 as depicted in Figure 1 of Socrates et al. (2011). For example, a Jupiter analogue starting at an orbital period $P \approx 12$ yrs that circularizes to a final distance $a_F \approx 0.06$ AU with $e_0 \approx 0.1$ migrates in a time t_m given by

$$t_m = t(e_i = 0.994) - t(e_0 = 0.1) \approx \frac{4}{3} t_D. \quad (34)$$

For a fixed value of J or a_F , the migration time t_m does not vary considerably if the initial value of the orbital period varies at the order unity level. For high- e migration to operate the above expression requires that t_D satisfies

$$t_D \lesssim \frac{3}{4} t_0 \quad (35)$$

or the lag time obeys

$$\tau \gtrsim 0.66 \text{ s} \quad (36)$$

for a Jupiter analogue.

With our definition given by eq. 23 along with the value of τ above to place a limit on Q during high- e migration

$$Q_{\text{HEM}} \lesssim 1.1 \times 10^5. \quad (37)$$

Note that t_m and consequently, τ and Q_{HEM} are rather insensitive to e_i .

3.4. discussion

If indeed Jupiter analogues with $P_F = 5$ days successfully undergo high- e migration, then according to the results of §§3.1 and 3.3, *there is at least an order of magnitude disagreement between the two requirements*. In other words, the time τ required for high- e migration is ten times larger than the allowed upper-limit for τ of Jupiter. For example, if Jupiter's lag time $\tau_{\text{J-Io}} = 0.66$ s, as required for high- e migration, then its tidal quality factor would decrease by an order of magnitude such that $Q_{\text{J-Io}} \lesssim 6 \times 10^3$.

The tension between two requirements for τ is obscured when tidal dissipation is quantified by Q . At face value, the requirements for Q from circularization and high- e migration indicate that less energy is dissipated per cycle than that inferred from the Jupiter-Io calibration. However, as detailed in §2.3, the tidal Q , unlike the lag time τ , is not an intrinsic property of the forced body in the equilibrium theory. Consequently, if the determination of τ from the Jupiter-Io interaction is correct, then Table 1 reveals that a Jupiter analogue would not be able to migrate from its initial position to $P_F \approx 5$ days within the Hubble time.

Note that the requirement on τ from tidal circularization of hot Jupiters is marginally consistent with the value inferred from the Jupiter-Io interaction, which is roughly in agreement with several previous studies (e.g., Leconte et al. 2010; Hansen 2010 & 2012).

4. PREVIOUS WORK ON HIGH-ECCENTRICITY MIGRATION

The fact that the lag time τ necessary to migrate a Jupiter analogue to a final circularization period $P_F \approx 5$

days is at least 10 times larger than τ_{J-Io} does not present an enormous hurdle for high- e migration scenarios. Due to the strong dependence of t_D on the final circularization radius a_F and planet radius R_p , it is possible that a factor of 10 may be recovered in order to make the migration time t_m consistent with $t_0 \lesssim 10$ Gyr.

However, there is a general understanding in the literature that for modest values of tidal dissipation inferred from the Jupiter-Io interaction ($Q_{J-Io} \approx \text{a few} \times 10^5$), the migration time of a Jupiter analogue with $P_F \approx 5$ days is of order ~ 300 Myrs (Wu & Murray 2003; Fabrycky & Tremaine 2007) in stark contrast with our results of the last section.

4.1. Wu & Murray 2003

Wu & Murray (2003; hereafter WM) produced the first calculations of high- e migration for hot Jupiters, with particular emphasis on HD 80606b.² Their evolutionary equations are taken from Eggleton et al. (1998). In this model, tidal dissipation is ultimately modeled after the orbit-averaged constant time lag model of Hut (1981). In order to determine the dissipation rate, a constant time lag τ must be specified.

However, inspection of their eq. (A9) reveals that the value of τ in their calculations is

$$\tau_{WM} = \frac{1}{nQ_0} = 192.8 \left(\frac{3 \times 10^5}{Q_0} \right) \left(\frac{P}{12 \text{ yrs}} \right) \text{ s} \quad (38)$$

where n is the mean motion and they set $Q_0 = 3 \times 10^5$. Note that τ_{WM} is not a constant during evolution and may be ~ 3000 times larger than upper limit for τ_{J-Io} given by Table 1. This particular deformation of the constant τ model of equilibrium tides results in an infinite amount of energy dissipated per orbit in the limit that $e \rightarrow 1$ and $P \rightarrow \infty$ for a fixed periastron. Finally, in this $e \rightarrow 1$ limit, the tidal $Q_{HEM} = -2\pi E_0 / \Delta E \rightarrow 0$.

Similar errors can be found in related works (Wu et al. 2007; Wu & Lithwick 2010). For example, consider eq. 5 of Wu et al. (2007), which they state is equivalent to eq. 10 of Hut (1981), in fact reveals that

$$\left. \frac{d \ln e}{dt} \right|_{Wu} \sim \left. \frac{d \ln e}{dt} \right|_{Hut} \times \frac{P}{P_F}. \quad (39)$$

Since the orbit is pseudo synchronized during high- e migration, P_F , J and a_F are all fixed during evolution. Therefore, for a orbit starting with $P = 12$ yrs migrating on a track of angular momentum J corresponding to $P_F = 5$ days, the lhs of the expression above exceeds the rhs by a factor $\sim P/P_F \sim 10^3$.

4.2. Fabrycky & Tremaine 2007

Like WM, Fabrycky & Tremaine (2007; hereafter FT) employ the evolutionary equations of Eggleton et al. (1998), where the prescription for tidal dissipation is based upon Hut (1981). Unlike WM however, they did not alter the orbit evolutionary equations. They qualitatively reproduce the presumed migration history of HD

80606b calculated by WM. In order to do so, they used a viscous time t_V of

$$t_{V_{FT}} = 0.001 \text{ yr} = 8.8 \text{ hrs.} \quad (40)$$

The qualitative agreement between WM and FT seems to strengthen the case for high- e migration with respect to the history of HD 80606b. When computing the distribution of the inclined hot Jupiters, FT use a value for t_V of

$$t_{V_{FT}} = 0.01 \text{ yr} = 88 \text{ hrs.} \quad (41)$$

However, while FT used the correct equations of motion their value for t_V departs from the value determined from the Jupiter-Io interaction given in Table 1

$$t_{V_{J-Io}} = 1.3 \times 10^5 \text{ hrs,} \quad (42)$$

by four orders of magnitude. Furthermore, note that the value FT use for t_V in both cases leads to a value of $|\Delta E/E_0| \approx 0.1$ during high- e migration.

In addition, FT write an expression in their equation A10 for the tidal Q that follows

$$Q \propto P \quad (43)$$

which as previously discussed at length, is incorrect within the context of high- e migration.

4.3. discussion of previous work

The mis-calibrations highlighted above demonstrate the danger of using Q when parameterizing the strength of tidal dissipation. For the case of the synchronization tide in the Jupiter-Io system, there is little quantitative difference between using Q and τ since the forcing period of Jupiter is approximately constant through the evolution of Io's orbit. Consequently, in the equations of motion, replacing lag time – or phase lag – in the equations of motion can be done without loss of accuracy e.g., as in Goldreich & Soter (1966). Nevertheless, due to the manner in which Q is defined, its placement within equation of motion (cf. §II of Goldreich & Soter 1966) can be extremely confusing or incorrect altogether.

The intuition derived from Goldreich & Soter (1966) for the synchronization tide of Jupiter cannot be applied to the more general tidal problem, where the orbit is eccentric. In the case of the Jupiter-Io interaction, a phase lag and physical lag angle are equal to one another, both of which are equivalent to the energy dissipated over a well-defined periodic sinusoidal forcing cycle. There is, in general, no well-defined forcing cycle in the tidal problem and therefore, no single phase lag. Furthermore, for the example of a pseudo synchronized eccentric orbit, the lag angle is a function of orbital phase, where its value at e.g., apoastron and periastron are not equal.

Despite previous mis-calibrations in the level of tidal dissipation in migrating super-eccentric gas giants, as previously mentioned, Table 1 indicates that a modest increase by a factor of ~ 10 in the relative strength of dissipation allows for Jupiter analogues to migrate from a period of 12 years to 5 days within the age of the Universe. This is a direct consequence of the fact that tidal dissipation is an extremely strong function of orbital angular momentum J , periastron r_p or a_F . For example, $a_F = 0.071$ AU in Figure 1 of FT, rather than HD 80606b's value for $a_F = 0.061$ AU, dissipation is also intrinsically weaker since they chose $M_p = 7.8 M_J$, migration was completed in $\sim 3 - 4$ Gyrs, rather than $t_0 = 10$

² In a planetary context, high- e migration was first utilized to explain the misalignment of Triton's orbit with the spin of Neptune by Goldreich et al. (1989). Those authors do not provide justification for their prescription for tidal dissipation. In the high- e limit, the behavior of their evolutionary equations is, correctly, identical to that of Hut (1981).

Gyrs, at long orbital periods the fractional amount of time migrating was small since the planet underwent Kozai oscillations and therefore, spent a significant fraction of its history at high- J . By accounting for all these effects, well over two of the orders that separate eqs. 40 and 42 may be erased. In what follows, we discuss some of the consequences for studies of gas giants that result from the requirement of modestly enhanced dissipation for high- e migration.

5. SOME IMPLICATIONS FOR STUDIES OF GAS GIANTS

5.1. *tidal theory*

It was realized early on that turbulent convection is severely inadequate in providing the required amount of dissipation in order to account for Jupiter's tidal Q (Goldreich & Nicholson 1977). Since then, the leading candidate for the source of tidal dissipation has been the resonant excitation of low frequency, nearly incompressible, waves that are primarily restored by the Coriolis force. This topic is technically challenging and continues to be in a state of development (Ogilvie & Lin 2004; Arras 2004; Wu 2005a,b; Goodman & Lackner 2009; Ogilvie 2009).

The results and relations of equilibrium tide theory cannot be directly compared with predictions of dynamical tidal theories. Nevertheless, the value of ΔE required for circularization for a given P_F within the Hubble time is largely independent of any particular tidal model. Goodman & Lackner (2009) write an expression for a tidal $Q = -\Delta E/2\pi E_0$ during hot Jupiter circularization that results from the inertial wave model. In their expression for Q , their peak energy E_0 is the peak energy in the fluctuating component of the interaction energy, appropriate for circularization given by our eq. A14 in the Appendix.

Their reported value for Q is $Q \gtrsim 3 \times 10^7$ (e.g., Goodman & Lackner 2009) for Jupiter analogues (core radius $R_c \lesssim 0.2R_J$) with $P_F = 5$ days. Table 1 indicates that the required Q for circularization of hot Jupiters with $P_F \approx 5$ days is $Q \leq 3 \times 10^5$. Thus, calculations of tidal circularization of gas giant planets arising from the inertial wave mechanism fall short by a factor of 100 or so. High- e migration will then be almost certainly more difficult to accomplish by the inertial wave mechanism. However, a calculation of the forced response that leads to the inertial wave mechanism for a highly eccentric orbit has yet to be performed, so a quantitative comparison with results from equilibrium tidal theory are therefore, not yet possible.

A clear difference between the properties of the tidal potential felt by gas giants undergoing high- e migration and Jupiter is the amplitude of the tidal potential. The ratio is given by $U_{\text{to}}/U_{\star} \approx 10^{-3}$. If indeed super-eccentric migrating Jupiters experience enhanced tidal dissipation during migration, then it may be due to non-linear effects.

5.2. *the inflated radii problem*

The hot Jupiters are inflated in that, unlike Jupiter, their radii are, in general, significantly larger than the zero temperature solution. There are several solutions to this problem, many of which depend upon tidal dissipation in one way or the other.

There is a class of mechanisms that can be thought of “delayed” contraction mechanisms. Here, the planet is brought into its close orbit in such a way that its degeneracy is lifted (Burrows et al. 2007; Leconte et al. 2010; Ibgui & Burrows 2009; Batygin & Stevenson 2010; see also Wu & Lithwick 2011). One candidate is tidal dissipation due to circularization during high- e migration. Table 1 indicates that in order for these mechanisms to work, tidal dissipation must be much stronger for super-eccentric migrating Jupiters in comparison to that inferred from the Jupiter-Io interaction.

Thermal tidal torques can render the hot Jupiters in a persistent state of asynchronous spin. The resulting on-going dissipation of the gravitational tide, presumably at great depth, allows for a steady source of thermal energy required to inflate the radii significantly above the zero temperature solution (Arras & Socrates 2009a, 2009b; 2010). In order for this mechanism to work, the out-of-phase quadrupole induced by thermal forcing near the optical photosphere must be equal to the out-of-phase quadrupole induced by dissipation of the gravitational tide. Such an equilibrium can only be realized if the relative level of tidal dissipation in the hot Jupiters is comparable to that inferred from the Jupiter-Io interaction. Therefore, if high- e migration is indeed responsible for delivering gas giants from initial orbital periods of ≈ 12 yrs to a final circularized state of ≈ 5 days, then thermal tidal torques cannot compete against the gravitational tidal torques. All of this, however, is under the assumption that the lag time for the synchronization and circularization tide are equal, which may not be the case in reality.

6. SUMMARY

If high- e migration leads to the production of hot Jupiters with final circularized orbital periods $P_F \approx 5$ days, tidal dissipation in these extrasolar gas giants must be at least 10 times stronger than that inferred from the Jupiter-Io interaction. While this alone cannot rule out high- e migration, it is in severe conflict with several previous studies of high- e migration. We showed that the discrepancy between our analysis and these studies arises from the fact that they employed dissipation strengths which are $10^3 - 10^4$ larger than the inferred upper limit of the Jupiter-Io interaction.

While current theories of tidal excitation and dissipation in gas giant planets can account for the inferred level of dissipation in the Jupiter-Io interaction (Ogilvie & Lin 2004) they cannot currently account for the amount of dissipation required for the observed circularization of hot Jupiters. If high- e migration is in fact responsible for hot Jupiter migration, the discrepancy between the level of computed dissipation in the inertial wave mechanism and that required to account for migration is even larger. If, for example, thermal emission from migrating Jupiters in orbit about late-type stellar hosts are observed by on-going and future direct-imaging campaigns (Dong et al. 2012), then the tidal dissipation time t_D cannot be much larger than $t_D \lesssim \text{Gyr}$.

In the Appendix we point out that the tidal quality factor is well-defined and can be related to a phase lag in only in two limiting cases: asynchronous and circular orbits or pseudo-synchronous orbits with small eccentricity. We find the relation between the tidal quality factor Q

and the lag time τ for these two limiting cases previously mentioned and additionally during high- e migration.

Given the results presented here, there is now tension between various sub-fields of gas giant studies: the theory of tidal friction in gas giant planets, few-body dynamics of exoplanet systems, theory of inflated radii, the evolutionary scenario outlined by Goldreich & Soter (1966) for the Jupiter-Io interaction and even perhaps planet formation. Either the standard thinking in one (or more) of these categories requires revision or there are extra ingredients at work in forming (stellar spin-orbit) mis-aligned hot Jupiters. The production of hot Jupiters is not a solved problem, though high- e migration is still viable.

We thank Phil Arras and Scott Tremaine for many

helpful discussions. We thank Daniel Fabrycky and Yanqin Wu for comments and suggestions on the manuscript. We also thank Jeremy Goodman, Dong Lai, Yoram Lithwick, John Papaloizou, Eliot Quataert, and Matias Zaldarriagga for stimulating discussions. Work by SD was performed under contract with the California Institute of Technology (Caltech) funded by NASA through the Sagan Fellowship Program. B.K. was supported by NASA through Einstein Postdoctoral Fellowship awarded by the Chandra X-ray Center, which is operated by the Smithsonian Astrophysical Observatory for NASA under contract NAS8-03060. AS acknowledges support from a John N. Bahcall Fellowship awarded by the Institute for Advanced Study, Princeton.

REFERENCES

- Arras, P. 2004, Presented at the KITP Program: Planet Formation: Terrestrial and Extra Solar, Feb 10, 2004, Kavli Institute for Theoretical Physics, University of California, Santa Barbara,
 Arras, P., & Socrates, A. 2009, arXiv:0901.0735
 Arras, P., & Socrates, A. 2009, arXiv:0912.2318
 Arras, P., & Socrates, A. 2010, ApJ, 714, 1
 Burrows, A., Hubeny, I., Budaj, J., & Hubbard, W. B. 2007, ApJ, 661, 502
 Blaes, O., Lee, M. H., & Socrates, A. 2002, ApJ, 578, 775
 Dong, S., Katz, B., & Socrates, A. 2012, arXiv:1201.4399
 Dong, S., Katz, B., & Socrates, A. 2012, arXiv:1204.1056
 Eggleton, P. P., Kiseleva, L. G., & Hut, P. 1998, ApJ, 499, 853
 Fabrycky, D., & Tremaine, S. 2007, ApJ, 669, 1298
 Lynden-Bell, D., & Ostriker, J. P. 1967, MNRAS, 136, 293
 Goldreich, P., Murray, N., Longaretti, P. Y., & Banfield, D. 1989, Science, 245, 500
 Goldreich, P., & Nicholson, P. D. 1977, Icar, 30, 301
 Goldreich, P., & Soter, S. 1966, Icar, 5, 375
 Goodman, J., & Lackner, C. 2009, ApJ, 696, 2054
 Hut, P. 1981, A&A, 99, 126
 Hansen, B. M. S. 2010, ApJ, 723, 285
 Hansen, B. M. S. 2012, ApJ, 757, 6
 Ibgui, L., Burrows, A., & Spiegel, D. S. 2010, ApJ, 713, 751
 Leconte, J., Chabrier, G., Baraffe, I., & Levrard, B. 2010, A&A, 516, A64
 Miller, M. C., & Hamilton, D. P. 2002, ApJ, 576, 894
 Nagasawa, M., Ida, S., & Bessho, T. 2008, ApJ, 678, 498
 Newcomb, W. A. 1962, Nuclear Fusion: Supplement Part 2 (Vienna: International Atomic Energy Agency), 45
 Ogilvie, G. I., & Lin, D. N. C. 2004, ApJ, 610, 477
 Ogilvie, G. I. 2009, MNRAS, 396, 794
 Rasio, F. A., & Ford, E. B. 1996, Science, 274, 954
 Reisenegger, A. 1994, ApJ, 432, 296
 Socrates, A., Katz, B., Dong, S., & Tremaine, S. 2012, ApJ, 750, 106
 Socrates, A., & Katz, B., 2012, ArXiv pre-prints
 Winn, J. N., Fabrycky, D., Albrecht, S., & Johnson, J. A. 2010, ApJ, 718, L145
 Wu, Y. 2005, ApJ, 635, 674
 Wu, Y. 2005, ApJ, 635, 688
 Wu, Y., & Murray, N. 2003, ApJ, 589, 605
 Wu, Y., Murray, N. W., & Ramsahai, J. M. 2007, ApJ, 670, 820
 Wu, Y., & Lithwick, Y. 2011, ApJ, 735, 109
 *John N. Bahcall Fellow
 † Einstein Fellow
 †† Sagan Fellow

APPENDIX

INTERACTION POTENTIAL, LOVE NUMBER, PEAK ENERGY AND ENERGY TRANSFER RATE

We review the mathematical machinery necessary to address the tidal problem. We assume the forced object is perturbed by a source of gravity of mass m at distance $D(t)$ in a Keplerian orbit with true anomaly $\Phi(t)$ and that the strength of the tidal field is weak. Therefore, only terms that are responsible for linear perturbing forces and linear responses are kept. The main purpose of what follows is to provide the necessary physical ingredients that are required to connect the results of equilibrium tidal theories that utilize a constant time lag model for tidal dissipation (Hut 1981) with those models that utilize a tidal quality factor Q (cf. Goldreich & Soter 1966).

interaction potential

To leading order in perturbation theory, the interaction potential is given by (cf. Newcomb 1962)

$$H_I = \int d^3x \rho \boldsymbol{\xi} \cdot \nabla U_T \quad (\text{A1})$$

where $\int d^3x$ is taken over the volume of the forced body, $\rho(\mathbf{x})$ is its static fluid density and $\boldsymbol{\xi}(\mathbf{x}, t)$ is the Lagrangian displacement field. The tidal potential U_T is given by

$$\begin{aligned} U_T &= -Gm \sum_{\ell m} \frac{4\pi}{2\ell+1} \left(\frac{r^\ell}{D^{\ell+1}} \right) Y_{\ell m}^*(\pi/2, \Phi) Y_{\ell m}(\boldsymbol{\Omega}) \\ &= - \sum_{\ell m} r^\ell Y_{\ell m}(\boldsymbol{\Omega}) \Psi_{\ell m}(t) \end{aligned} \quad (\text{A2})$$

which serves to define $\Psi_{\ell m}(t)$. Note that we have specialized to the case where the perturber is located at co-latitude $\theta' = \pi/2$. Now, we may write the interaction energy as

$$H_I = - \sum_{\ell m} q_{\ell m}^* \Psi_{\ell m} \quad (\text{A3})$$

where $q_{\ell m}$ is the multipole moment (cf. Press & Teukolsky 1977)

$$q_{\ell m}(t) = \int d^3x \rho \boldsymbol{\xi} \cdot \nabla r^\ell Y_{\ell m}^*(\boldsymbol{\Omega}). \quad (\text{A4})$$

Note that for the $\ell = 2$, the $\Psi_{\ell m}$'s carry dimensions of frequency squared.

k_ℓ – the Love number

The Love number k_ℓ is defined as

$$U_{\text{ind}}^\ell(R) = k_\ell U_T^\ell(R) \quad (\text{A5})$$

where U_{ind}^ℓ is the additional gravitational potential resulting for the induced tidal response of degree ℓ and R is the un-perturbed radius of the forced body. The induced potential is given by (e.g., Jackson 1999)

$$U_{\text{ind}} = -G \sum_{\ell m} \frac{4\pi}{2\ell + 1} q_{\ell m} r^{-\ell-1} Y_{\ell m}(\boldsymbol{\Omega}) \quad (\text{A6})$$

and with this we can solve for each $q_{\ell m}$ in terms of k_ℓ , $\Psi_{\ell m}$ and R

$$q_{\ell m} = \Psi_{\ell m} / \varpi_\ell^2 \quad (\text{A7})$$

where

$$\varpi_\ell^2 \equiv \frac{4\pi G}{k_\ell (2\ell + 1) R^{2\ell+1}}. \quad (\text{A8})$$

E_0 – the peak energy

The tidal Q is a dimensionless number that parameterizes the relative level of dissipation of the forced body over some interval in time. Given a tidal theory, a solution of the equations of motion allows for a determination of the energy dissipated ΔE over a forcing cycle. In order to construct the dimensionless parameter Q , which is equivalent to a phase lag, a characteristic energy E_0 is needed. Typically, E_0 is referred to as the “peak energy.” It is equal to the average energy in the tidal interaction or the peak energy of the potential energy of the flow. When computing E_0 only fluctuating components of the tidal potential are considered in order to maintain the standard relationship between Q and the phase lag. In the literature, E_0 is defined in many ways and for the sake of clarity, we provide a definition for it.

Insert the expression for the induced multipoles in terms of k and $\Psi_{\ell m}$ from eq. A7 into the expression for H_I and obtain

$$H_I = - \sum_{\ell m} k_\ell \frac{2\ell + 1}{4\pi} \frac{|\Psi_{\ell m}|^2 R^{2\ell+1}}{G} = - \sum_{\ell m} |\Psi_{\ell m}|^2 / \varpi_\ell^2. \quad (\text{A9})$$

The interaction energy H_I has contributions from both time-independent and fluctuating components. We define the peak energy E_0 with the following expression

$$E_0 \equiv - \langle H_I \rangle = \frac{1}{P_{\text{cyc}}} \oint_{\text{cyc}} dt \sum_{\text{osc}, \ell m} |\Psi_{\ell m}^2| / \varpi_\ell^2 = \sum_{\text{osc}, \ell m} |\Psi_{\ell m}^2| / \varpi_\ell^2 \quad (\text{A10})$$

where the integral is taken over the forcing cycle and only terms resulting from the oscillating component of the tidal potential are kept.

synchronization: $e = 0$, $\Omega \neq \Omega_{\text{ps}}$

Tidal synchronization for circular orbits is due to forcing from the $\ell = 2$ and $m = \pm 2$ portions of the tidal potential, which are responsible for its time-varying component. In this case, the oscillating component of the tidal field can be thought of as being due to a single Fourier harmonic with forcing frequency $\sigma = 2|n - \Omega|$. The peak energy in the interaction E_0 is then given by

$$E_0 = - \langle H_I \rangle = \sum_{\ell=2, m=\pm 2} |\Psi_{\ell m}|^2 / \varpi_\ell^2$$

$$E_0 = \frac{3 k_\ell G m^2 R^5}{4 a^6}. \quad (\text{A11})$$

During synchronization, there is an $m = 0$ tidal deformation as well. However, since it does not oscillate, it cannot contribute to energy transfer and consequently, does not factor into the definition of E_0 .

circularization: $e \ll 1$, $\Omega = \Omega_{\text{ps}}$

For slightly eccentric orbits with $e \ll 1$ in pseudo synchronous spin states with $\Omega \approx n$, isolation of the oscillating terms of U_T can be accomplished by a Fourier decomposition. In the rotating frame, let

$$\left(\frac{a}{D}\right)^{\ell+1} e^{im\Phi} = \sum_{k=-\infty}^{\infty} X_k^{\ell m}(e) e^{i\sigma_{km}t} \quad (\text{A12})$$

where the forcing frequency $\sigma_{km} = kn - m\Omega$ and $X_k^{\ell m}(e)$ are the Hansen coefficients, which are given by

$$\begin{aligned} X_k^{\ell m}(e) &= \frac{n}{2\pi} \int_0^{2\pi/n} dt e^{im\Phi - i\sigma_{km}t} \left(\frac{a}{D}\right)^{\ell} \\ &\simeq \delta_{km} + \frac{e}{2} [(\ell+1+2m)\delta_{k,m+1} + (\ell+1-2m)\delta_{k,m-1}] + \mathcal{O}(e^2) \end{aligned} \quad (\text{A13})$$

to leading order in eccentricity. In the $e \ll 1$, $\Omega = \Omega_{\text{ps}} \approx n$ limit, the only terms that contribute to time-dependent driving are, for $\ell = 2$, the $(m = 0, k = 1)$, $(m = 2, k = 2 \pm 1)$ and the $(m = -2, k = -2 \pm 1)$ terms. Note that all of the oscillation frequencies are therefore given by $\sigma_{km} = \pm n$.

With decomposition into Hansen coefficients, the interaction energy H_I and peak energy E_0 becomes

$$\begin{aligned} E_0 &= -\langle H_I \rangle = \sum_{\text{osc}, \ell m k} |\Psi_{\ell m k}|^2 / \varpi_{\ell}^2 \\ E_0 &= \frac{21 e^2 k_2}{2} \frac{G m^2 R^5}{a^6} \end{aligned} \quad (\text{A14})$$

where the $\Psi_{\ell m k}$'s are defined by the relation

$$\Psi_{\ell m} = G m W_{\ell m} \frac{e^{im\Phi}}{D^{\ell+1}} = \sum_k \frac{G m W_{\ell m}}{a^{\ell+1}} X_k^{\ell m}(e) e^{i\sigma_{km}t} \equiv \sum_k \Psi_{\ell m k} e^{i\sigma_{km}t} \quad (\text{A15})$$

and the Fourier transform of the induced multipole moments $q_{\ell m k}$ are

$$q_{\ell m} = \sum_k q_{\ell m k} e^{i\sigma_{km}t} \quad (\text{A16})$$

each of which satisfy

$$q_{\ell m k} = \Psi_{\ell m k} / \varpi_{\ell}^2. \quad (\text{A17})$$

high- e migration: $e \rightarrow 1$, $\Omega = \Omega_{\text{ps}}$

For the case of high- e migration, the time dependance of the tidal potential is far from that of a single Fourier harmonic. Its magnitude is highly peaked at periastron and falls off quickly with increasing distance. We choose to define the peak energy as the entire interaction potential, since all of the $Y_{\ell m}$'s vary with time and therefore, contribute to the potential energy available to perform work on the orbit. We write

$$E_0 \equiv \max[-H_I] = k_2 \frac{G m^2 R^5}{r_p^6} \quad (\text{A18})$$

where $r_p = a(1 - e)$ is the periastron distance.

energy transfer rate

The rate of energy transfer \dot{E} between the orbit and internal degrees of freedom of the forced body is given by

$$\dot{E} = - \int d^3x \rho \dot{\xi} \cdot \nabla U_T. \quad (\text{A19})$$

The definition of U_T in terms of the $\Psi_{\ell m}$'s may be employed in order to write

$$\dot{E} = - \sum_{\ell m} \dot{q}_{\ell m}^* \Psi_{\ell m}. \quad (\text{A20})$$

CONDITIONS FOR A SINGLE Q FROM MANY OSCILLATORS AND THE TIDAL PROBLEM

Even for the case of semi-diurnal forcing ($\ell = 2; m = \pm 2$) during synchronization of a body in a circular orbit, the forcing may be thought of as originating from two driving terms. It is not clear that the level of tidal dissipation in such a system can be summarized by a single Q , which itself is composed of a single phase lag. In order to proceed, a definition for Q is required for a set of decoupled forced damped harmonic oscillators, each of which obey

$$\ddot{X}_n + \omega_n^2 X_n + \gamma_n \dot{X}_n = F_n(t) = \frac{F_{0,n}}{\sqrt{2}} e^{i\sigma_n t} + \text{c.c.} \quad (\text{B1})$$

where ω_n is the fundamental frequency of each oscillator amplitude X_n , γ_n is the corresponding damping rate and σ_n is the driving frequency. Each of the n oscillators has a $Q = Q_n$ given by

$$\frac{1}{Q_n} = -\frac{\oint_{\text{cyc}} dE_n/dt}{2\pi E_{0,n}} = -\frac{-\oint_{\text{cyc}} \dot{X}_n F_n}{2\pi \langle X_n F_n \rangle} = \sigma_n \tau_n = \delta_n \quad (\text{B2})$$

where $\tau_n = \gamma_n/\omega_n^2$ is the lag time and δ_n is the phase lag. In the expression above $X_n(t)$ is the solution in the non-resonant weak friction limit where $X_n(t) = F_n(t - \tau_n)/\omega_n^2$.

Now, define Q for the entire set of oscillators

$$\frac{1}{Q} \equiv -\frac{\sum_n \oint_{\text{cyc}} dE_n/dt}{\sum_n 2\pi E_{0,n}} = -\frac{\sum_n \oint_{\text{cyc}} dE_n/dt}{\sum_n 2\pi \langle X_n F_n \rangle} = \frac{\sum_n \delta_n |F_{0,n}|^2 / \omega_n^2}{\sum_n |F_{0,n}|^2 / \omega_n^2} \quad (\text{B3})$$

The expression above is well-defined only if there is a single forcing cycle and therefore, only a single forcing frequency. In the event that there is a single lag time for each oscillator, then $\delta_n = \delta$ is a constant. It follows that

$$\frac{1}{Q} = \delta_n = \delta. \quad (\text{B4})$$

Or in other words, a single Q that is equivalent to a single phase lag quantifies the level of dissipation of a set decoupled oscillators only if their lag times are equal to one another.

To make contact with the tidal problem, we restrict to $\ell = 2$ and write the tidal Q as

$$\frac{1}{Q} = -\frac{\oint_{\text{cyc}} dE/dt}{2\pi E_0} = \frac{\oint_{\text{cyc}} dE/dt}{2\pi \langle H_I \rangle} = \frac{\oint_{\text{cyc}} dE/dt}{\sum_{\ell=2,m} 2\pi \langle q_{\ell m}^* \Psi_{\ell m} \rangle} = \frac{\sum_{\ell=2,m} \oint_{\text{cyc}} \dot{q}_{\ell m}^* \Psi_{\ell m}}{\sum_{\text{osc}, \ell=2,m} 2\pi q_{\ell m}^* \Psi_{\ell m}}. \quad (\text{B5})$$

As in the case of a set of decoupled forced non-resonant weakly damped harmonic oscillators, a single Q for the combined system only makes sense if there is a single tidal forcing frequency. The time derivative $\dot{q}_{\ell m}^*$ enforces that the $\sum_{\ell m}$ in the numerator runs over only the oscillating components of the tidal potential and the corresponding tidal response. Expand into Fourier harmonics and write

$$\frac{1}{Q} = \frac{\sum_{\ell=2,mk} \delta_{\ell mk} |\Psi_{\ell mk}|^2 / \varpi_\ell^2}{\sum_{\text{osc}, \ell=2,mk} |\Psi_{\ell mk}|^2 / \varpi_\ell^2}. \quad (\text{B6})$$

Each individual phase lag $\delta_{\ell mk} = \sigma_{mk} \tau_\ell$ are equal to one another in the event that their respective lag times are equal to one another as well. If this is the case, we may then write

$$\frac{1}{Q} = \delta_{\ell mk} = \delta. \quad (\text{B7})$$

Comparison of eq. B6 with eq. B3 shows a correspondence between a set of non-resonantly forced, weakly damped harmonic oscillators (X_n, F_n) with the tidal problem ($q_{\ell m}, \Psi_{\ell m}$) in the equilibrium weak-friction limit. Though the general tidal problem is not equivalent to a set of decoupled harmonic oscillators with constant time lags, their respective expressions for Q are equivalent under these limited conditions. That is, a single tidal Q that represents a single phase lag only holds when $e \ll 1$ and $\Omega = n$ for circularization or $e = 0$ during synchronization. In addition, the time lag for each of the m components of the tidal force of degree $\ell = 2$ must be equal to one another. From this, we arrive at the following conclusion: *a single Q can serve to parameterize the level of tidal dissipation only if there is a single forcing frequency σ and a single constant lag time τ .* That is, parameterization of tidal dissipation with a constant τ is applicable in a more general class of orbital conditions in comparison to parameterization with a tidal Q .